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Complete Continuity of the Free Gas Scattering Operator in Neutron Thermalization Theory

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1. INTRODUCTION

Let us consider the integrodifferential equation describing the motion of neutrons in an absorbing and scattering homogeneous medium:

$$\frac{\partial n(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v} \cdot \operatorname{grad}_{\mathbf{r}} n(\mathbf{r}, \mathbf{v}, t) = - v\Sigma_t(v)\, n(\mathbf{r}, \mathbf{v}, t) + \int_{\omega_{\mathbf{v}'}} v'\Sigma_s(\mathbf{v}' \to \mathbf{v})\, n(\mathbf{r}, \mathbf{v}', t)\, d\omega_{\mathbf{v}'} + q(\mathbf{r}, \mathbf{v}, t) \tag{1}$$

where $n(\mathbf{r}, \mathbf{v}, t)$ is the distribution function of the neutrons in terms of the variables of space and velocity, $\mathbf{r}$ and $\mathbf{v}$, at the time t, $v'\Sigma_s(\mathbf{v}' \to \mathbf{v})$ is the scattering rate per neutron from velocity $\mathbf{v}'$ to velocity $\mathbf{v}$, $v\Sigma_t(v)$ the total collision rate per neutron at velocity $\mathbf{v}$, $q(\mathbf{r}, \mathbf{v}, t)$ is a source term and, finally, $\omega_{\mathbf{v}}$ $(\omega_{\mathbf{v}'})$ denotes the three-dimensional space of the vectors $\mathbf{v}$ $(\mathbf{v}')$.

In this paper we will focus our attention on the collision operators $v\Sigma_t \cdot$, $\int v'\Sigma_s(\mathbf{v}' \to \mathbf{v}) \cdot d\omega_{\mathbf{v}'}$, and study some properties of these, and related operators, within the scheme of the free gas model. We assume that the nuclei of the scattering medium have a velocity distribution of the Maxwell-Boltzmann type, and that scattering collisions between the neutrons and the nuclei can be respresented as collisions between hard spheres; for the sake of simplicity, we assume also that the scattering cross section of a nucleus, say σ_s (i.e., π times the square of "the sum of the radii of the colliding hard spheres"), is a constant, independent of the relative velocity of the two particles before the collision, and, finally, that the total absorption rate per neutron is independent of v. This means that the macroscopic absorption

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cross section varies according to the $1/v$ law and, consequently, we can write

$$v\Sigma_t(v) = \int_{\omega_{\mathbf{v}'}} v\Sigma_s(\mathbf{v} \rightarrow \mathbf{v}')\, d\omega_{\mathbf{v}'} + \gamma \tag{2}$$

where γ is a constant.

Under these assumptions it has been shown [1-3][1] that the functions $v\Sigma_t(v)$ and $v'\Sigma_s(\mathbf{v}' \rightarrow \mathbf{v})$ can be written in terms of well known trascendental functions. Namely, if a new unknown function is introduced

$$\varphi(\mathbf{r}, \mathbf{v}, t) = e^{\beta^2 v^2/2}\, n(\mathbf{r}, \mathbf{v}, t) \tag{3}$$

(where $\beta = \sqrt{(m/2kT)}$, m being the neutron mass, k the Boltzmann constant, T the temperature of the medium), and the new independent variables

$$\mathbf{p} = \beta \mathbf{v} \tag{4}$$

$$\tau = \beta^{-1} t \tag{5}$$

are used, then Eq. (1) becomes

$$\frac{\partial\varphi(\mathbf{r}, \mathbf{p}, \tau)}{\partial\tau} + \mathbf{p} \cdot \operatorname{grad}_r \varphi(\mathbf{r}, \mathbf{p}, \tau) = -h(p)\,\varphi(\mathbf{r}, \mathbf{p}, \tau) + \int_{\omega_{\mathbf{p}'}} H(\mathbf{p}, \mathbf{p}')\,\varphi(\mathbf{r}, \mathbf{p}', \tau)\, d\omega_{\mathbf{p}'} + s(\mathbf{r}, \mathbf{p}, \tau) \tag{6}$$

where

$$s(\mathbf{r}, \mathbf{p}, \tau) = \beta e^{-p^2/2} \cdot q\left(\mathbf{r}, \frac{\mathbf{p}}{\beta}, \tau\beta\right),$$

and the aforementioned collision rates assume the form

$$h(p) = N\sigma_s \left[\frac{e^{-\mu p^2}}{\sqrt{\pi\mu}} + \left(p + \frac{1}{2\mu p}\right) \operatorname{erf}(\sqrt{\mu} p)\right] + \beta\gamma \tag{7}$$

$$H(\mathbf{p}, \mathbf{p}') = H(\mathbf{p}', \mathbf{p}) = N\sigma_s \frac{(\mu + 1)^2}{4(\pi\mu)^{3/2}} \cdot \frac{1}{\rho} \exp\left\{-\tfrac{1}{4}\left[\frac{\rho^2}{\mu} + \frac{\mu}{\rho^2}(p^2 - p'^2)^2\right]\right\}. \tag{8}$$

[1] The last paper is a monography dealing with the theory of gases, but a proper extension of the arguments of pp. 30-35, 65-70 leads immediately to the same results as the preceding more specialized papers.

Here $\rho = |\mathbf{p} - \mathbf{p}'|$ is the modulus of the (dimensionless) relative velocity, $\mu = M/m \geqslant 1$ is the ratio of the mass M of a nucleus to the mass of a neutron, and N is the number of nuclei per unit volume.

2. Some Preliminary Results

Let us first study the function $h(p)$. We have

$$\lim_{p\to 0} h(p) = \frac{2N\sigma_s}{\sqrt{\pi\mu}} + \beta\gamma$$

$$\lim_{p\to\infty} \frac{1}{p} h(p) = N\sigma_s$$

$$h'(p) = N\sigma_s \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\mu}p} \left(1 - \frac{t^2}{\mu p^2}\right) e^{-t^2} dt > 0.$$

Thus $h(p)$ is a positive, monotonically increasing function for $0 \leqslant p < \infty$, with $h(p) \to +\infty$ of the order of p as $p \to +\infty$.

Let $L^2(\omega_{\mathbf{p}})$ denote the Hilbert space of the complex valued measurable functions which are defined and square summable on $\omega_{\mathbf{p}}$. Since $1/h(p)$ is bounded, the operator

$$Mf \equiv \frac{1}{h(p)} f(\mathbf{p}), \qquad f(\mathbf{p}) \in L^2(\omega_{\mathbf{p}}) \tag{9}$$

is obviously a bounded multiplication operator in $L^2(\omega_{\mathbf{p}})$.

Next we prove the following inequality:

$$I = \int_{\omega_{\mathbf{p}'}} H^{\xi}(\mathbf{p}, \mathbf{p}') \frac{1}{(1+p')^{\eta}} d\omega_{\mathbf{p}'} < \frac{C}{(1+p)^{\eta+1}} \qquad (1 \leqslant \xi < 3; 0 \leqslant \eta) \tag{10}$$

which has already been obtained, in the particular case $\mu = 1$, by T. Carleman [3]. Let θ be the angle between $\mathbf{p}$ and $\mathbf{p} - \mathbf{p}'$. Then

$$p'^2 = p^2 + \rho^2 + 2p\rho\cos\theta$$

and our integral can be written

$$I = N\sigma_s \frac{(\mu+1)^{2\xi}}{4^{\xi}(\pi\mu)^{3\xi/2}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{\rho^{\xi}}$$

$$\cdot \frac{\exp\{-(\xi/4)[(\rho^2/\mu) + \mu(\rho + 2p\cos\theta)^2]\}}{(1 + \sqrt{p^2 + \rho^2 + 2p\rho\cos\theta})^{\eta}} \rho^2 \sin\theta \, d\rho \, d\theta \, d\chi.$$

By integrating on the azimuthal angle χ and setting

$$x = \frac{\rho}{p} + 2\cos\theta$$

$$y = \frac{\rho}{p}$$

we get

$$I = N\sigma_S \frac{(\mu+1)^{2\xi}}{4^{\xi}(\pi\mu)^{2\xi/2}} \cdot \frac{\pi}{p^{\xi-3}} \iint_D \frac{\exp\{-(\xi p^2/4\mu)(\mu^2x^2+y^2)\}}{y^{\xi-2}(1+p\sqrt{1+xy})^{\eta}}\, dxdy$$

where D denotes the domain

$$|x-y| \leqslant 2, \qquad 0 \leqslant y < \infty$$

Let D_1 be the half-ellipse

$$\mu^2x^2 + y^2 < \tfrac{1}{4}, \qquad y > 0$$

which for $\mu \geqslant 1$ is completely contained in D, and let $D_2 = D - D_1$. We write, accordingly,

$$I = I_1 + I_2 .$$

Then

$$I_1 < \frac{C_1}{p^{\xi-3}} \iint_{D_1} \frac{\exp\{-(\xi p^2/4\mu)(\mu^2x^2+y^2)\}}{y^{\xi-2}(1+p\sqrt{1-(1/4\mu)})^{\eta}}\, dxdy$$

$$< \frac{C_2}{p(1+p\sqrt{1-(1/4\mu)})^{\eta}} \int_{-\sqrt{\xi/\mu}(p/4)}^{\sqrt{\xi/\mu}(p/4)} dt \int_0^{\sqrt{\xi/\mu}(p/4)} \frac{e^{-(t^2+z^2)}}{z^{\xi-2}}\, dz$$

$$< \frac{C_3}{p(1+p\sqrt{1-(1/4\mu)})^{\eta}}$$

$$I_2 < \frac{C_4}{p^{\xi-3}} \iint_{D_2} \exp\left(-\frac{\xi p^2}{8\mu}\cdot\frac{1}{4}\right) \frac{\exp[-(\xi p^2/8\mu)(\mu^2x^2+y^2)]}{y^{\xi-2}}\, dxdy$$

$$< \frac{C_4}{p^{\xi-3}} e^{-\xi p^2/32\mu} \int_0^{\infty} \frac{e^{-(\xi p^2/8\mu)y^2}}{y^{\xi-2}}\, dy \int_{y-2}^{y+2} dx < C_5 e^{-\xi p^2/32\mu} \int_0^{\infty} \frac{e^{-t^2}}{t^{\xi-2}}\, dt$$

$$< C_6 e^{-\xi p^2/32\mu}$$

where all the C's are independent of $\mathbf{p}$. Thus

$$I < \frac{C_3}{p(1+ap)^{\eta}} + C_6 e^{-bp^2}$$

where a, b are some constants, with $0 < a < 1$, $b > 0$. But I cannot become infinite as $p \to 0$ since

$$I < C_7 \int_0^\infty \frac{1}{\rho^{\xi-2}} e^{-(\xi/4\mu)\rho^2} d\rho < C_8 < \infty$$

and we conclude that a constant $C > 0$ exists such that

$$I < \frac{C}{(1+p)^{\eta+1}} \qquad \text{Q.E.D.}$$

3. The Kernels $H(\mathbf{p}, \mathbf{p}')$ and $S(p, p')$

Now we consider the kernel $H(\mathbf{p}, \mathbf{p}')$ of the scattering operator of Eq. (6), and denote by $H_n(\mathbf{p}, \mathbf{p}')$ its nth iterate. We can prove that $H_3(\mathbf{p}, \mathbf{p}')$ is square summable. In fact, by the aid of the Schwarz inequality and using (10) we have

$$\begin{aligned} H_2(\mathbf{p}, \mathbf{p}') &= \int_{\omega_{\mathbf{p}''}} H(\mathbf{p}, \mathbf{p}'')\, H(\mathbf{p}'', \mathbf{p}')\, d\omega_{\mathbf{p}''} \\ &\leqslant \left\{ \int_{\omega_{\mathbf{p}''}} H^2(\mathbf{p}, \mathbf{p}'')\, d\omega_{\mathbf{p}''} \right\}^{1/2} \left\{ \int_{\omega_{\mathbf{p}''}} H^2(\mathbf{p}'', \mathbf{p}')\, d\omega_{\mathbf{p}''} \right\}^{1/2} \\ &< \frac{C_2'}{(1+p)^{1/2}(1+p')^{1/2}}; \end{aligned}$$

then

$$\begin{aligned} H_3(\mathbf{p}, \mathbf{p}') &= \int_{\omega_{\mathbf{p}''}} H(\mathbf{p}, \mathbf{p}'')\, H_2(\mathbf{p}'', \mathbf{p}')\, d\omega_{\mathbf{p}''} \\ &\leqslant \frac{C_2'}{(1+p')^{1/2}} \int_{\omega_{\mathbf{p}''}} H(\mathbf{p}, \mathbf{p}'') \frac{d\omega_{\mathbf{p}''}}{(1+p'')^{1/2}} < \frac{C_3'}{(1+p)^{3/2}(1+p')^{1/2}} \end{aligned}$$

and, making use of the symmetry of $H_3(\mathbf{p}, \mathbf{p}')$,

$$H_3(\mathbf{p}, \mathbf{p}') < \frac{C_3'}{(1+p)(1+p')}.$$

By the same way we obtain estimates for the successive iterates:

$$H_n(\mathbf{p}, \mathbf{p}') < \frac{C_n'}{(1+p)^{(n-1)/2}(1+p')^{(n-1)/2}}. \tag{11}$$

It follows:

$$H_6(\mathbf{p}, \mathbf{p}) = \int_{\omega_{\mathbf{p}}'} H_3^2(\mathbf{p}, \mathbf{p}')\, d\omega_{\mathbf{p}'} < \frac{C_6'}{(1+p)^5}$$

and finally

$$\int_{\omega_{\mathbf{p}}}\int_{\omega_{\mathbf{p}}'} H_3^2(\mathbf{p}, \mathbf{p}')\, d\omega_{\mathbf{p}} d\omega_{\mathbf{p}'} < 4\pi C_6' \int_0^\infty \frac{p^2 dp}{(1+p)^5} < C_7' < \infty \qquad \text{Q.E.D.}$$

Let H denote the integral symmetric operator whose kernel is $H(\mathbf{p}, \mathbf{p}')$. Since

$$\int_{\omega_{\mathbf{p}}'} H(\mathbf{p}, \mathbf{p}')\, d\omega_{\mathbf{p}'} \leqslant C \int_0^\infty \rho e^{-\rho^2/4\mu} d\rho < C'$$

we have

$$\left| \int_{\omega_{\mathbf{p}}}\int_{\omega_{\mathbf{p}}'} H(\mathbf{p}, \mathbf{p}') f(\mathbf{p}) \overline{g(\mathbf{p}')}\, dw_{\mathbf{p}'} d\omega_{\mathbf{p}} \right|$$

$$\leqslant \tfrac{1}{2} \int_{\omega_{\mathbf{p}}}\int_{\omega_{\mathbf{p}}'} H(\mathbf{p}, \mathbf{p}') \left[| f(\mathbf{p}) |^2 + | g(\mathbf{p}') |^2 \right] d\omega_{\mathbf{p}'} d\omega_{\mathbf{p}}$$

$$\leqslant \frac{C'}{2} \int_{\omega_{\mathbf{p}}} | f(\mathbf{p}) |^2\, d\omega_{\mathbf{p}} + \frac{C'}{2} \int_{\omega_{\mathbf{p}}} | g(\mathbf{p}) |^2\, d\omega_{\mathbf{p}}$$

for any two arbitrarily chosen functions $f, g \in L^2(\omega_{\mathbf{p}})$. Then H is bounded ([4], p. 56). Furthermore, the operator H^3 has a square summable kernel and is, therefore, completely continuous. By a known theorem ([5], p. 317), this implies that H itself is completely continuous, and we have proven the following:

THEOREM I. *$Hf \equiv \int H(\mathbf{p}, \mathbf{p}') f(\mathbf{p}')\, d\omega_{\mathbf{p}'}$ is a completely continuous symmetric operator in $L^2(\omega_{\mathbf{p}})$, with a square summable third iterate kernel.*

In many problems H operates on functions which are dependent only on the modulus p of the velocity. Then, when performing the integration on $d\omega_{\mathbf{p}} = p^2 dp d\Omega_{\mathbf{p}}$, one can first integrate over the angles and obtain (we omit the tedious but straightforward calculation)

$$p'^2 \int_{4\pi} H(\mathbf{p}, \mathbf{p}')\, d\Omega_{\mathbf{p}'} = R(p, p')$$

where

$$R(p, p') = \frac{p'}{p} S(p, p'),$$

$$S(p, p') = S(p', p) = N\sigma_s \frac{(\mu + 1)^2}{4\mu} \Big\{ e^{(p^2 - p'^2)/2} \Big[\operatorname{erf}\Big(\frac{\mu+1}{2\sqrt{\mu}} p - \frac{\mu-1}{2\sqrt{\mu}} p'\Big) \pm \operatorname{erf}\Big(\frac{\mu+1}{2\sqrt{\mu}} p + \frac{\mu-1}{2\sqrt{\mu}} p'\Big)\Big]$$

$$+ e^{(p'^2 - p^2)/2} \Big[\operatorname{erf}\Big(\frac{\mu+1}{2\sqrt{\mu}} p' - \frac{\mu-1}{2\sqrt{\mu}} p\Big) \mp \operatorname{erf}\Big(\frac{\mu+1}{2\sqrt{\mu}} p' + \frac{\mu-1}{2\sqrt{\mu}} p\Big)\Big]\Big\} \tag{12}$$

(the upper signs refer to $p < p'$ and the lower signs to $p > p'$). Now, it is easily proven that

$$p'^2 \int_{4\pi} H_n(\mathbf{p}, \mathbf{p}')\, d\Omega_{\mathbf{p}'} = R_n(p, p') = \frac{p'}{p} S_n(p, p')$$

where $R_n(p, p')$, $S_n(p, p')$ denote the nth iterates of $R(p, p')$, $S(p, p')$. Then we have, from (11),

$$S_6(p, p') < pp' \frac{4\pi C_6'}{(1 + p)^{5/2} (1 + p')^{5/2}}$$

and finally

$$\iint_0^\infty S_3^2(p, p')\, dp dp' = \int_0^\infty S_6(p, p)\, dp < 4\pi C_6' \int_0^\infty \frac{p^2 dp}{(1 + p)^5} < \infty.$$

Let us consider the Hilbert space $L^2(0, \infty)$ of the functions which are square summable on $0 \leqslant p < \infty$. The kernel $S(p, p')$ individuates an integral operator S in this space. S is bounded, since

$$\Big| \int_0^\infty \int_0^\infty S(p, p') f(p) \overline{g(p')}\, dp dp' \Big|$$

$$= \Big| \int_{\omega_{\mathbf{p}}} \int_{\omega_{\mathbf{p}'}} H(\mathbf{p}, \mathbf{p}') \frac{f(p)}{\sqrt{4\pi}\, p} \frac{\overline{g(p')}}{\sqrt{4\pi}\, p'}\, d\omega_{\mathbf{p}} d\omega_{\mathbf{p}'} \Big| < \infty$$

where

$$\int_0^\infty | f(p) |^2\, dp = \int_{\omega_{\mathbf{p}}} \frac{| f(p) |^2}{4\pi p^2}\, d\omega_{\mathbf{p}} < \infty$$

$$\int_0^\infty | g(p) |^2\, dp = \int_{\omega_{\mathbf{p}}} \frac{| g(p) |^2}{4\pi p^2}\, d\omega_p < \infty.$$

By the same arguments which led to Theorem I we obtain:

THEOREM II. *$Sf \equiv \int S(p, p') f(p')\, dp'$ is a completely continuous symmetric operator in $L^2(0, \infty)$, with a square summable third iterate kernel.*

This completes our analysis.

4. THE STATIONARY, SPACE INDEPENDENT EQUATION

As an application of the preceding results, let us consider the following equation:

$$h(p)\, \varphi(\mathbf{p}) - \int_{\omega_{\mathbf{p}'}} H(\mathbf{p}, \mathbf{p}')\, \varphi(\mathbf{p}')\, d\omega_{\mathbf{p}'} = s(\mathbf{p}) \tag{13}$$

which is a particular case of Eq. (6). The question is now if this equation obeys Fredholm theory, provided that some assumptions are made on the domain in which the solutions are to be founded. We stipulate that $q(p)$ and $s(p)$ must be in $L^2(\omega_{\mathbf{p}})$. Then, by dividing both members by $h(p)$ and using (9), Eq. (13) may be written

$$\varphi - MH\varphi = Ms \tag{14}$$

Owing to the complete continuity of H and the boundedness of M the operator $K = MH$ is completely continuous ([5], p. 315). Thus Fredholm theory actually holds for Eq. (14) or (13). Next, let us suppose that $s(\mathbf{p}) \equiv s(p)$. Then also $\varphi(\mathbf{p}) \equiv \varphi(p)$ and, by performing the integration upon the angles and putting $\psi(p) = p\varphi(p)$, $z(p) = ps(p)$, we obtain the following equation

$$h(p)\, \psi(p) - \int_0^\infty S(p, p')\, \psi(p')\, dp' = z(p) \tag{15}$$

which is of frequent use in physics [6]. Since

$$\int_0^\infty |\, z(p)\, |^2\, dp = \frac{1}{4\pi} \int_{\omega_{\mathbf{p}}} |\, s(p)\, |^2\, d\omega_{\mathbf{p}} < \infty$$

by the same way as before we can see, without any further assumptions, that also Eq. (15) obeys Fredholm theory.

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